Optimal stopping for the multidimensional Brownian motion

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Part I: The optimal stopping problem (OSP)

- Given
  - (A) a stochastic process $X = \{X_t \colon t \geq 0\}$ taking values in $\mathcal{E}$
  - (B) a discount rate $r \geq 0$,
  - (C) a payoff function $g(x) \colon \mathcal{E} \to [0, \infty)$.
- Find:
  - (I) $V(x)$, the *value function*,
  - (II) τ^*, the *optimal stopping time*,

  such that

$$\boxed{V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x\, e^{-r\tau} g(X_\tau) = \mathbf{E}_x\, e^{-r\tau^*} g(X_\tau^*)}$$

Technical details

- There is an underlying measurable space $(\Omega, \mathcal{F})$, a filtration $\{\mathcal{F}_t\}$, and a family of probabilities $\{\mathbf{P}_x\}$, such that X is a continuous time Markov process taking values in $\mathbb{R}$.

- $\mathcal{M} = \{\tau\}$ is the class of all stopping times:

$$\tau\colon \Omega \to [0, \infty], \ \{\tau \le t\} \in \mathcal{F}_t \ \forall t.$$

A first example

(A) The stochastic process is Brownian motion $\{B_t\}$ on $\mathbb{R}$.

(B) The discount rate is $r = 2$.

(C) The payoff function is the identity

$$g(x) = x^+ = \max(x, 0),$$

The optimal stopping problem is:

Find $V(x)$, τ^* s.t.

$$\boxed{V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}\, e^{-2\tau}(x + B_\tau)^+ = \mathbf{E}\, e^{-2\tau^*}(x + B_{\tau^*})^+}$$

Solution

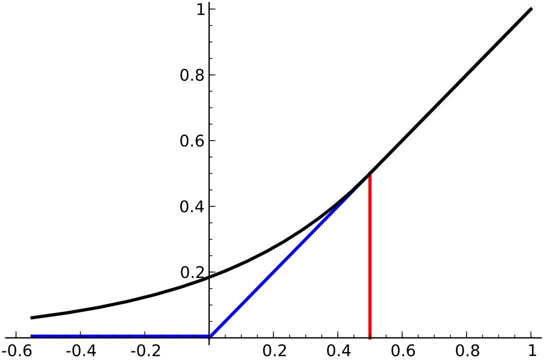
The solution was found by M.H. Taylor (1968, Annals of Math. Statistics). Denoting $x^* = 1/2$, it is given by

$$V(x) = \begin{cases} \frac{1}{2}e^{2x-1}, & \text{if } x < x^*. \\ x, & \text{if } x \geq x^*, \end{cases}$$

$$\tau^* = \inf\{t \geq 0 \colon x + B_t \geq 1/2\}$$

- The point x^* is the *critical threshold*, and determines τ^*,

Value function



- The payoff function $g(x)$ (data)
- The value function $V(x)$ (solution)
- The optimal threshold $x^* = 1/2$ (solution)

The problem can be formulated as an ODE with moving boundary. The *differential operator* associated with the process is

$$L^X f = \frac{1}{2} f'',$$

We are solving the problem

$$\begin{cases} L^X f = \frac{1}{2} f'' \leq rf, \\ f \geq g, \\ (L^X f - rf)(f - g) = 0. \end{cases}$$

Here $g(x) = x^+$.

Part II: General optimal stopping problems

- Given

  (A) a stochastic process $X = \{X_t \colon t \geq 0\}$ taking values in $\mathcal{E}$

  (B) a discount rate $r \geq 0$,

  (C) a payoff function $g(x) \colon \mathcal{E} \to [0, \infty)$.

- Find:

  (I) $V(x)$, the *value function*,

  (II) τ^*, the *optimal stopping time*,

  such that

$$\boxed{V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x\, e^{-r\tau} g(X_\tau) = \mathbf{E}_x\, e^{-r\tau^*} g(X^*_\tau)}$$

Our approach

When considering optimal stopping we typically find two classes of results:

- **Concrete problems:** explicit solutions to concrete optimal stopping problems (fixed X and g)
- **Theoretical results:** for wide classes of processes and reward functions.

Verification approach

- Usually in concrete problems, one –somehow– guesses the solution
- Apply the *smooth fit principle*: solve the *continuous fit* equation and the *smooth fit* equation.
- Finally, prove that this guess in fact solves the optimization problem.
- This approach, when an explicit solution can be found, is very effective. Details in Chapter IV of Peskir and Shiryaev (2006).

Theoretical results

Relevant initial results are

- the super-martingale characterization of the solution by Snell (1952)
- the Markovian characterization by Dynkin (1963): V is the least excessive function, that is larger or equal than g.

Representation approach

- We consider X as a Markov process
- We represent V as an integral of the Green function with respect to $\mu(dx)$.

$$\boxed{V(x) = \int_{\mathcal{E}} G_r(x,y)\mu(dy) = \int_S G_r(x,y)\mu(dy)}$$

- the support of μ is the stopping region S of the problem, giving the optimal stopping rule τ^*.

Main assumptions

$X = \{X_t \colon t \geq 0\}$ is a standard Markov process

- The infinitesimal generator L is defined by

$$L\phi(x) = \lim_{h \to 0} \frac{\mathbf{E}_x\, \phi(X_t) - \phi(x)}{h}$$

- The Green kernel of the process is defined by

$$G_r(x, dy) = \int_0^\infty e^{-rt}\, \mathbf{P}_x(X_t \in dy)dt.$$

- We assume the existence of a measure $m(dy)$ such that

$$G_r(x, dy) = G_r(x, y)m(dy).$$

Main ingredients:

- Dynkin's characterization of the value function: V is the least r-excessive function such that $V(x) \geq g(x)$ for all x
- Riez's representation of an r-excessive functions V:

$$V(x) = \int_{\mathcal{E}} G_r(x, y)\mu(dy) + (r\text{-harmonic function}) \qquad \text{(R)}$$

- Inversion formula: the infinitesimal generator and the resolvent are inverse operators, for a test function ϕ

$$\phi(x) = \int_{-\infty}^{\infty} G_r(x, y)(r - L)\phi(y)m(dy) \qquad \text{(IF)}$$

Key properties - Riesz Representation R

- As the harmonic term provides constant reward, the representation (R) becomes

$$V(x) = \int_{\mathcal{E}} G_r(x,y)\mu(dy).$$

- As V is harmonic on A if and only if $\mu(A) = 0$, and V is harmonic on the continuation region C, we have

$$V(x) = \int_S G_r(x,y)\mu(dy)$$

where S is the stopping region.

This approach was initiated by Salminen (1985) with the Martin Kernel, see also M.-Salminen (2007)

Summarizing

We have obtained

$$V(x) = \int_S G_r(x,y)\mu(dy), \tag{R}$$

$$V(x) = \int_S G_r(x,y)(r-L)V(y)m(dy). \tag{IF}$$

We further know that $g = V$ in S. Under the assumption

$$(r-L)V(x) = (r-L)g(x) \quad \text{for } x \in S$$

for $x \in S$ we conclude that the representing measure is

$$\boxed{\mu(dy) = (r-L)g(y)m(dy) \quad \text{in } S.}$$

Remaining problem: find S

Optimal stopping for diffusions in $\mathbb{R}$

- Functions ψ y φ are the fundamental solutions of the equation
$$Lf = rf,$$
where L is the infinitesimal generator.

- The Green kernel satisfies
$$G_r(x,y) = \begin{cases} w_r^{-1}\psi(x)\varphi(y), & x \le y; \\ \\ w_r^{-1}\psi(y)\varphi(x), & x \ge y. \end{cases}$$
where w_r is a norming constant (the *Wronskian*)

To find $S = [x^*, \infty)$ we solve the equation:

$$g(x^*) = w_r^{-1} \int_{x^*}^{\infty} \psi(x^*)\varphi(y)(r - L)g(y)m(dy).$$

For the Brownian motion, life is simple:

$$G(x, y) = e^{-\sqrt{2r}|x-y|},$$

because

$$\psi(x) = e^{\sqrt{2r}x}, \qquad \varphi(x) = e^{-\sqrt{2r}x}.$$

Based on this approach, we solve problems where the solution is not differentiable at x^*. [1]

[1] Crocce-M. *Stochastics* (2013)

Part III: Multidimensional optimal stopping

We have now $r > 0$, a multidimensional Wiener process

$$X = (W^1, \dots, W^d)$$

and a function

$$g \colon \mathbb{R}^d \to \mathbb{R},$$

and want to solve the problem

$$\boxed{V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \, e^{-r\tau} g(X_\tau) = \mathbf{E}_x \, e^{-r\tau^*} g(X^*_\tau)}$$

Comments

- In fact, $g(X_t)$ is a one-dimensional process, but in general it is not Markovian.
- If $g(x) = g(Ax)$ where A is a rotation, then it is Markovian. In this case, the problem can be solved with one-dimensional techniques.
- The PDE formulation is

$$\begin{cases} L^X f = \frac{1}{2}\Delta f \leq rf, \\ f \geq g, \\ (L^X f - rf)(f - g) = 0. \end{cases}$$

Representation approach

Theorem[2]

(a) The continuation set C is the solution to the *Green Kernel equation*:

$$\int_C G_r(x,y)(r-L)g(x)dx = 0,$$

for all $x \in S$.

(b) The continuation set C is the solution to the *Martin Kernel equation*:

$$\boxed{\int_C e^{ax}(r-L)g(x)dx = 0,}$$

for all $a \in \mathbb{R}^d$ s.t. $\|a\|^2 = 2r$, $\{h(x) = e^{ax}\}$ is the set of r-harmonic functions.

(c) The set C is unique in a convenient class of sets.

[2] Joint work with S. Christensen, F. Crocce and P. Salminen

Example

$$X = (W^1, W^2), \qquad g(x,y) = x^2 + \alpha^2 y^2.$$

If $\alpha = 1$ the process $g(X(t))$ is a Bessel(2) process, and the problem transforms in a one-dimensional problem. If $\alpha \neq 1$ it is a “true” multidimensional problem.
We have an infinitesimal generator

$$Lf = \frac{1}{2}\Delta f,$$

The function V satisfies

- $LV = rV$ in the continuation region C
- $g = V$ in the stopping region $S = C^c$
- C should be found

Solving the *Martin Kernel equation*

We have for each a an equation, and the unknown is the set C (or its boundary).

$$\int_C e^{ax}(r - L)g(x)dx = 0,$$

By super-martingale arguments we know that $C \supset N$, where

$$N = \{x \in \mathbb{R}^d : (r - L)g(x) \leq 0\},$$

is the *negative set*.

Discretization

So we take affine polar coordinates to determine C, and discretize the problem: Given N, denoting

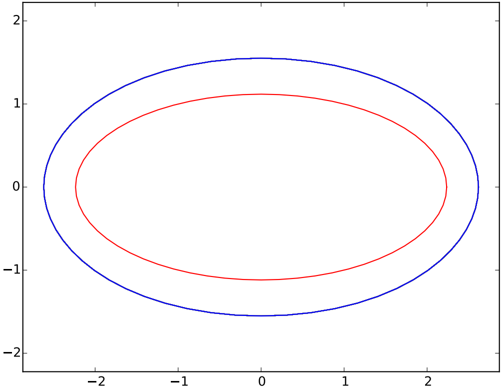
$$\theta_i = i/(2\pi N) \ (0 \leq i < N)$$

we determine the unknowns $\rho_i = \rho(\theta_i)$ with conditions $a_j \sim \phi_j = j/(2\pi N) \ (0 \leq j < N)$:

$$\sum_{i=0}^{N-1} F_2(\rho_i, \gamma(\theta_i, \phi_j)) = \sum_{i=0}^{N-1} F_1(\gamma(\theta_i, \phi_j)), \quad 0 \leq i, j < N.$$

We plot the solution for $\alpha = 2$ and $r = 1$ and $r = 0.3$, considering $N = 64$.

Solution



- The red set is $\{(r - L)g(x) \leq 0\}$ the negative set
- The blue line delimitates the optimal stopping region

Example: 3 dimensional problem

- The process is a three dimensional Brownian motion

$$X = (W^1, W^2, W^3),$$

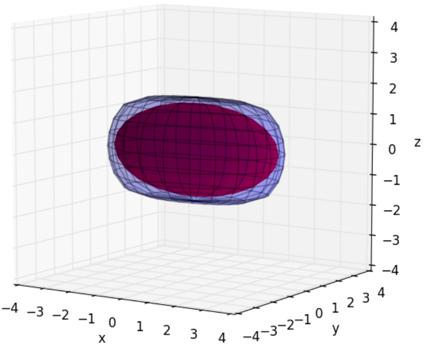
- The reward function is quadratic:

$$g(x, y, z) = x^2 + \alpha^2 y^2 + \beta^2 z^2.$$

(For illustration we take $\alpha = 2,\ \beta = 3,\ r = 1$).

- The computations are similar, with spherical coordinates.

Solution



- The inner set is $\{(r - L)g(x) \leq 0\}$ the negative set
- The violet surface delimitates the optimal stopping region

¡Muchas gracias!